Slowly decaying tails of massive scalar fields in spherically symmetric spacetimes

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(Received 28 December 2001; published 8 April 2002)

We study the dominant late-time behaviors of massive scalar fields in static and spherically symmetric spacetimes. Considering the field evolution in the far zone where the gravitational field is weak, we show under which conditions the massive field oscillates with an amplitude that decays slowly as $t^{-5/6}$ at very late times, as previously found in (say) the Schwarzschild case. Our conclusion is that this long-lived oscillating tail is generally observed at timelike infinity in black hole spacetimes, while it may not be able to survive if the central object is a normal star. We also discuss that such a remarkable backscattering effect is absent for the field near the null cone at larger spatial distances.

DOI: 10.1103/PhysRevD.65.084031 PACS number(s): 04.20.Ex, 04.70.Bw

I. INTRODUCTION

One of the most remarkable features of wave dynamics in curved spacetimes is tails. Scalar, electromagnetic and gravitational fields in curved spacetimes do not, in general, propagate entirely along the null cone, but are accompanied by "tails" which propagate in the interior of the null cone. This implies that at late times waves do not cut off sharply but rather die off in tails.

In particular, it has been well established that the late-time evolution of massless scalar fields propagating in black-hole spacetimes is dominated by an inverse power-law behavior, as was first analyzed by Price [1]. In a brilliant work, Leaver [2] demonstrated that late-time tails can be associated with the existence of a branch cut in the Green's function for the wave propagation problem. Gundlach *et al.* [3] showed that power-law tails also characterize the late-time evolution of radiative fields at future null infinity, while the decay rate is different from that of timelike infinity. Furthermore, it has been shown that power-law tails are a genuine feature of gravitational collapse [4–6]: Late-time tails develop even when no horizon is present in the background, which means that power-law tails should be present in perturbations of stars, or after the implosion and subsequent explosion of a massless field which does not result in black hole formation. The existence of these tails was demonstrated in full nonlinear numerical simulations of the spherically symmetric collapse of a self-gravitational massless scalar field. Gundlach *et al.* [4] obtained the power-law tails for a massless field in fully nonlinear simulations at fixed r, Marsa and Choptuik [5] found them both at fixed r and along the event horizon, and Burko and Ori [6] at fixed r, the event horizon, and future null infinity to very high numerical accuracy.

When the scalar field has a nonzero mass, the tail behaviors are quite different from massless ones. For example, as is well known, the tails exist even in Minkowski spacetimes, which is related to the fact that different frequencies forming a massive wave packet have different phase velocities [7]. If the background spacetime is curved, it is expected that interesting features peculiar to massive fields develop through the scattering due to the spacetime curvature.

The important role of massive scalar fields has been revealed in elementary particle physics. For example, in higher-dimensional theories the Fourier modes of a massless scalar field behave like massive fields known as Kaluza-Klein modes, and the recent development of the Kaluza-Klein idea (e.g., the Randall-Sundrum model [8] in the string theory) strongly motivates us to understand the evolutional features due to the field mass in detail. In addition, scalar fields are astrophysically important, if boson stars made up of self-gravitating scalar fields prove to be viable candidates for dark matter [9]. If such astrophysical objects become unstable and collapse to form a black hole, both gravitational and scalar fields would be presumably radiated. Further, as one example of curious effects peculiar to massive fields, it has been argued that unstable quasi-normal modes can exist [10,11]. Then the time evolution of massive scalar fields in curved spacetimes (in particular, in black-hole spacetimes) would become an important problem to be solved.

Recently it was pointed out that the late-time tails of massive scalar fields in Reissner-Nordström spacetime are quite different from massless fields in the existence of the intermediate late time tails [12] (see also [13]). If the Compton wavelength m^{-1} of a massive field is much longer than the horizon radius of a black hole with the mass M, namely $mM \ll 1$, each multiple moment ψ of the field evolves into the oscillatory inverse power-law behavior

$$\psi \sim t^{-l-3/2}\sin(mt), \tag{1}$$

at intermediate late times. It is clear from Eq. (1) that massive fields decay slower than massless ones, and waves with peculiar frequency ω quite close to m mainly contribute to the massive tail, while the dominant contribution to massless tails should be evaluated in the zero-frequency limit $\omega \to 0$. Though the oscillatory power-law form (1) has been numerically verified at intermediate late times, $mM \ll mt \ll 1/(mM)^2$, it should be noted that the intermediate tails are not the final asymptotic behaviors; another wave pattern can dominate at very late times, when it still remains very difficult to determine numerically the exact decay rate [12,13]. In the previous paper [14], we have analytically found that the

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transition from the intermediate behavior to the asymptotic one occurs in a nearly extreme Reissner-Nordström background. The oscillatory inverse power-law behavior of the dominant asymptotic tail is approximately given by

$$\psi \sim t^{-5/6}\sin(mt), \tag{2}$$

independent of the multiple moment l, and the decay becomes slower than the intermediate ones. Then, the similar result for the decay rate has been obtained by considering massive scalar fields in Schwarzschild background (in the limited cases that $mM \ll 1$ or $mM \gg 1$, where M is the black-hole mass) [15] and massive Dirac fields in Kerr-Newman backgrounds [16]. Asymptotic behaviors of massive scalar fields in dilaton black-hole backgrounds have also been discussed [17].

These results given in [14–16] suggest that massive fields in black hole backgrounds decay as $t^{-5/6}$ generally at very late times. So it is an interesting subject to study how universally such a slowly decaying tail develops. It has been numerically shown [4] that a power-law tail develops even when the collapsing massless scalar field fails to produce a black hole. This is evidence for the late-time tail to be a direct consequence of wave scattering in far distant regions. In this paper we prove that the decay law $t^{-5/6}$ of massive scalar fields can be essentially determined by the analysis in the far zone where the gravitational field is weak. However, we can also derive the conditions for the tails with the decay rate of $t^{-5/6}$ to dominate as an asymptotic behavior. Considering the physical interpretation of the conditions, we can claim that any spherically symmetric black holes generate the same asymptotic tails, while the conditions may not be satisfied if the central object is a normal star.

In Sec. II we introduce the Green's function analysis to investigate the time evolution of a massive scalar field in any static, spherically symmetric spacetimes. In Sec. III we consider the approximation valid in the far zone, and we find the conditions for the tail with the decay rate of $t^{-5/6}$ to develop. The final section is devoted to discussion, which contains a comment that the tail behavior of $t^{-5/6}$ breaks down as the region comes close to the light cone. We discuss that the tail with the decay rate of $t^{-5/6}$ can develop also in rotating black hole spacetimes.

II. GREEN'S FUNCTION ANALYSIS

A. Massive scalar fields in spherically symmetric spacetimes

We consider the evolution of a massive scalar field in a static spherically symmetric background with the asymptotically flat metric given by

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

Here we do not assume the metric to be a solution of the vacuum or electrovac Einstein equations. The scalar field Φ with the mass m satisfies the wave equation

$$\Box\Phi = m^2\Phi. \tag{4}$$

Resolving the field into spherical harmonics

$$\Phi = \frac{\psi^l(t,r)}{r} Y_{l,m}(\theta,\varphi), \tag{5}$$

hereafter we omit the index l of ψ^l for simplicity, and we obtain a wave equation for each multiple moment

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V(r)\right]\psi = 0, \tag{6}$$

where r_* is the Wheeler tortoise coordinate defined by

$$\frac{dr_*}{dr} = \sqrt{\frac{h}{f}}, \tag{7}$$

and V is the effective potential

$$V = f\left[\frac{1}{r\sqrt{fh}}\left(\sqrt{\frac{f}{h}}\right)' + \frac{l(l+1)}{r^2} + m^2\right]. \tag{8}$$

The time evolution of the radial function ψ is given by

$$\psi(r_*,t) = \int [G(r_*,r'_*;t)\psi_t(r'_*,0) + G_t(r_*,r'_*;t)\psi(r'_*,0)]dr'_* \tag{9}$$

for $t \geq 0$, where the retarded Green's function G is defined as

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V\right]G(r_*,r'_*;t) = \delta(t)\delta(r_* - r'_*). \tag{10}$$

The causality condition requires that $G(r_*,r'_*;t) = 0$ for $t \leq 0$. In order to obtain $G(r_*,r'_*;t)$, we use the Fourier transform

$$\tilde{G}(r_*,r'_*;\omega) = \int G(r_*,r'_*;t)e^{i\omega t}dt, \tag{11}$$

which is analytic in the upper half ω plane. The corresponding inversion formula is

$$G(r_*,r'_*;t) = -\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}\tilde{G}(r_*,r'_*;\omega)e^{-i\omega t}d\omega, \tag{12}$$

where c is some positive constant. Now the Fourier component of the Green's function $\tilde{G}(r_*,r'_*;\omega)$ is expressed in terms of two linearly independent solutions for the homogeneous equation

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - V\right]\tilde{\psi}_i = 0, \quad i = 1,2. \tag{13}$$

The boundary condition for the basic solution $\tilde{\psi}_1$ is that it should be well behaved on the event horizon if the central object is a black hole, and at $r = 0$ otherwise. On the other hand, the other basic solution $\tilde{\psi}_2$ is required to be well be-

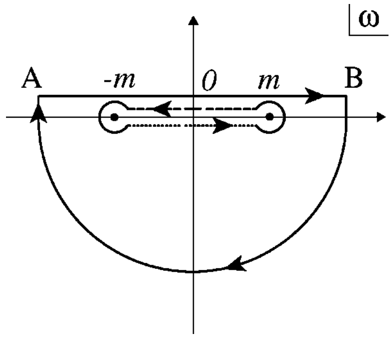

FIG. 1. The integration contour for Eq. (12), when $t > r_* - r'_*$. The original path corresponds to the straight line AOB. An integral along a branch cut placed along the interval $-m \leqslant \omega \leqslant m$ leads to the power-law tail.

haved at spatial infinity, $r \to \infty$. Using these two solutions, $\tilde{G}(r_*, r'_*; \omega)$ can be written by

$$\tilde{G}(r_*, r'_*; \omega) = \frac{1}{W(\omega)} \begin{cases} \tilde{\psi}_1(r'_*, \omega)\tilde{\psi}_2(r_*, \omega), & r_* > r'_*, \\ \tilde{\psi}_1(r_*, \omega)\tilde{\psi}_2(r'_*, \omega), & r_* < r'_*, \end{cases} \tag{14}$$

where $W(\omega)$ is the Wronskian defined by

$$W(\omega) = \tilde{\psi}_1 \tilde{\psi}_{2, r_*} - \tilde{\psi}_{1, r_*} \tilde{\psi}_2 . \tag{15}$$

The integrand in Eq. (12) has branch points at $\omega = \pm m$. Considering the branch points, one may change the integration path in Eq. (12). First, if $r_* - r'_* > t$, the path is closed in the upper half of the ω plane for the integration to converge. Since the integrand would have no singularities in the upper half plane, we obtain $G(r_*, r'_*; t) = 0$ according to the causality postulate. If $r_* - r'_* < t$, on the other hand, the path can be deformed to the curve shown in Fig. 1. As will be shown later, the late-time tails are generated owing to the existence of a branch cut (in $\tilde{\psi}_2$) placed along the interval $-m \leqslant \omega \leqslant m$.

B. The analysis in a region far from the gravitational source

It has been found in previous papers [14–17] that the oscillatory power-law tails of massive scalar fields whose decay rate is $t^{-5/6}$ dominate at asymptotically late times in black-hole spacetimes. In this paper we show that the decay law can be simply derived by considering wave modes only in a far distant region, as generic behaviors observed in any black hole spacetime.

For that purpose, we assume

$$\frac{r}{M} \gg 1, \tag{16}$$

where M is the gravitational mass of a background field. Then, the expansion of the metric functions f and h as a power series in M/r leads to

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + O(r^{-3}) \tag{17}$$

and

$$h = 1 + \frac{2M'}{r} + \frac{Q'^2}{r^2} + O(r^{-3}), \tag{18}$$

where M', Q and Q' are some parameters characterizing the background field in more detail in addition to the gravitational mass M. Expanding Eq. (13) in the same manner and neglecting terms of order $O[(M/r)^3]$ and higher, we obtain the approximated form

$$\frac{\partial^2 \tilde{\psi}}{\partial r^2} - U\tilde{\psi} = 0, \tag{19}$$

where

$$U = (m^2 - \omega^2) - \frac{2M\omega^2}{r} + \frac{2M'(m^2 - \omega^2)}{r} - \frac{\lambda^2 - \frac{1}{4}}{r^2}. \tag{20}$$

The coefficient λ in Eq. (20) depends on the multiple moment l and the other parameters M, M', Q and Q'. For example, in the case of the Reissner-Nordström background with mass M and charge Q, we have

$$\lambda = \sqrt{\left(l + \frac{1}{2}\right)^2 + 4m^2M^2 - 12\omega^2M^2 - m^2Q^2 + 2\omega^2Q^2}. \tag{21}$$

We keep the term of the order of $O[(M/r)^2]$ in Eq. (19), in order to confirm that the decay rate of asymptotic timelike tails found in [14–17] is independent of λ. Introducing the variable defined as

$$x = 2\varpi r, \tag{22}$$

where

$$\varpi = \sqrt{m^2 - \omega^2}, \tag{23}$$

Eq. (19) is rewritten by

$$\left[\frac{d^2}{dx^2} - \frac{1}{4} + \frac{\kappa}{x} - \frac{\lambda^2 - 1/4}{x^2}\right]\tilde{\psi} = 0, \tag{24}$$

where κ is

$$\kappa = \frac{Mm^2}{\varpi} - (M + M')\varpi. \tag{25}$$

III. TIMELIKE ASYMPTOTIC TAIL OF MASSIVE SCALAR FIELDS

A. The wave modes

Our aim is now to show that the tail with the power-law decay of $t^{-5/6}$ is a generic asymptotic behavior in black-hole spacetimes by using the wave modes satisfying Eq. (19). One may claim that the inner boundary condition to determine $\tilde{\psi}_1$ is missed if the analysis is limited to the range (16). Hence, we treat $\tilde{\psi}_1$ as a general solution for Eq. (19) and reveal a condition which allows the excitation of the asymptotic power-law tail. Fortunately we will be able to prove that such a condition is always satisfied if the event horizon exists in the background spacetime.

First, let us give $\tilde{\psi}_2$, by requiring that it damps exponentially for $|\omega|<m$ and is purely outgoing for $|\omega|>m$ at spatial infinity. The outer boundary condition leads to

$$\tilde{\psi}_2(\varpi,r)=W_{\kappa,\lambda}(x), \tag{26}$$

where $W_{\kappa,\lambda}(x)$ is the Whittaker function [18], and the branch of ϖ is chosen to be

$$\varpi=\begin{cases}\sqrt{m^2-\omega^2}, & \omega<m,\\ -i\sqrt{\omega^2-m^2}, & \omega>m.\end{cases} \tag{27}$$

Note that $W_{\kappa,\lambda}(x)$ is a many-valued function of ϖ, and there is a cut in $\tilde{\psi}_2$. The late-time tail is generated by the contribution from the branch cut in $\tilde{\psi}_2$, while $\tilde{\psi}_1$ is a one-valued function of ϖ, as was shown in [14,15]. This is because the late-time tail is a consequence of backscattering. On the other hand, we give $\tilde{\psi}_1$, using the Whittaker functions $M_{\kappa,\lambda}$ and free parameters a and b as follows,

$$\tilde{\psi}_1=aM_{\kappa,\lambda}(x)+bM_{\kappa,-\lambda}(x), \tag{28}$$

where a and b will be determined if the inner boundary condition for $\tilde{\psi}_1$ is specified. Nevertheless the relation

$$\tilde{\psi}_1(\varpi)=\tilde{\psi}_1(e^{i\pi}\varpi) \tag{29}$$

should be required, since $\tilde{\psi}_1$ is a one-valued function for ϖ. Then, we rewrite the parameters a and b as

$$a=j\varpi^{-1/2-\lambda} \tag{30}$$

and

$$b=k\varpi^{-1/2+\lambda}, \tag{31}$$

where j and k are some one-valued functions for ϖ. In the following we will clarify under what kind of conditions for a and b the power-law tail with the decay rate of $t^{-5/6}$ asymptotically dominates.

B. Branch cut integration at timelike asymptotic regions

As was shown in [12,14,15], late-time tails are derived by the integral of $\tilde{G}(r_*,r'_*;\omega)$ around the branch cut in Fig. 1. Using Eqs. (12), (26) and (28), the branch cut contribution to the Green's function is given by

$$\begin{aligned}G^C(r_*,r'_*;t)&=-\frac{1}{2\pi}\int_{-m}^{m}\tilde{\psi}_1(r')\left[\frac{\tilde{\psi}_2(r,\varpi)}{W(\varpi)}-\frac{\tilde{\psi}_2(r,e^{i\pi}\varpi)}{W(e^{i\pi}\varpi)}\right]e^{-i\omega t}d\omega\\&=\frac{1}{2\pi}\int_{-m}^{m}\tilde{\psi}_1(r')\left[\frac{ap_++bp_-}{ap_+-bp_-}-\frac{aq_++bq_-}{aq_+-bq_-}\right]\frac{\tilde{\psi}_1(r)}{4jk\lambda}e^{-i\omega t}d\omega,\end{aligned} \tag{32}$$

where

$$p_\pm=\frac{\Gamma(\pm2\lambda)}{\Gamma\left(\frac{1}{2}\pm\lambda-\kappa\right)} \tag{33}$$

and

$$q_\pm=\frac{\Gamma(\pm2\lambda)}{\Gamma\left(\frac{1}{2}\pm\lambda+\kappa\right)}e^{i\pi(1/2\mp\lambda)}. \tag{34}$$

Note that at very late times

$$mt\gg1, \tag{35}$$

the rapidly oscillating term $e^{-i\omega t}$ leads to a mutual cancellation between the positive and the negative parts of the integrand (32), except for the case that the other terms of the integrand also change rapidly with ω. In fact, it is easy to see that $\tilde{\psi}_1$ neither oscillates rapidly nor changes exponentially with ω in the region

$$\omega r\ll\kappa. \tag{36}$$

Then, if κ remains small, the effective contribution to the integral in Eq. (32) is claimed to be limited to the range $|\omega-m|=O(1/t)$ or equivalently $\varpi=O(\sqrt{m/t})$ (see [12,14,15]), and the intermediate tails become dominant at late times in the range

$$mM\ll mt\ll\frac{1}{(mM)^2}, \tag{37}$$

when the integral (32) should be estimated under the condition

$$\kappa\simeq\frac{m^2M}{\sqrt{m^2-\omega^2}}=O(mM\sqrt{mt})\ll1. \tag{38}$$

As was discussed in [14,15], the small value of κ represents that the backscattering due to the spacetime curvature is not effective at intermediate late times. It is obvious that the intermediate tails given by Eq. (1) dominate at intermediate late times (37), which was numerically supported by [12,13].

As was also discussed in [14,15], however, the intermediate tails cannot be an asymptotic behavior, and the long-term evolution from the intermediate behavior to the final one should occur. The asymptotic tail becomes dominant at very late times such that

$$mt \gg \frac{1}{m^2M^2}, \tag{39}$$

when the effective contribution to the integral (32) arises from the region

$$\kappa \simeq \frac{m^2M}{\sqrt{m^2-\omega^2}} \gg 1, \tag{40}$$

which means the backscattering effect due to the curvature-induced potential dominates. In the limit of $\kappa \to \infty$, the term $(ap_+ + bp_-)/(ap_+ - bp_-)$ becomes

$$\frac{ap_+ + bp_-}{ap_+ - bp_-} \to \frac{\eta_+ e^{i\pi\kappa} + \gamma_+ e^{-i\pi\kappa}}{\eta_- e^{-i\pi\kappa} + \gamma_- e^{i\pi\kappa}}, \tag{41}$$

which includes very rapid oscillations as $e^{\pm i\pi\kappa}$, and we have

$$\eta_\pm = \Gamma(2\lambda) a\kappa^{-\lambda} e^{-i\pi\lambda} \pm \Gamma(-2\lambda) b\kappa^{\lambda} e^{i\pi\lambda}, \tag{42}$$

and

$$\gamma_\pm = \Gamma(2\lambda) a\kappa^{-\lambda} e^{i\pi\lambda} \pm \Gamma(-2\lambda) b\kappa^{\lambda} e^{-i\pi\lambda}. \tag{43}$$

Such rapidly oscillatory behaviors are not seen in the other term $(aq_+ + bq_-)/(aq_+ - bq_-)$, which is given by

$$\frac{aq_+ + bq_-}{aq_+ - bq_-} \to \frac{a\Gamma(2\lambda)\kappa^{-\lambda}e^{-i\pi\lambda} + b\Gamma(-2\lambda)\kappa^{\lambda}e^{i\pi\lambda}}{a\Gamma(2\lambda)\kappa^{-\lambda}e^{-i\pi\lambda} - b\Gamma(-2\lambda)\kappa^{\lambda}e^{i\pi\lambda}} \tag{44}$$

even in the limit of $\kappa \to \infty$.

Now we revisit the procedure through which the branch cut integration (32) leads to the tail with the decay rate of $t^{-5/6}$, as was shown in [14,15]. For example, in the case of small mass field ($mM \ll 1$) in a Schwarzschild background with mass M, $\eta_\pm$ and $\gamma_\pm$ become

$$\eta_\pm = \frac{\Gamma(2\lambda)^2\Gamma(1-4i\omega M)}{\Gamma(1/2+\lambda-2i\omega M)^2}(4m^2M^2)^{-\lambda}e^{-i\pi\lambda} \pm \frac{\Gamma(-2\lambda)^2\Gamma(1-4i\omega M)}{\Gamma(1/2-\lambda-2i\omega M)^2}(4m^2M^2)^{\lambda}e^{i\pi\lambda} \tag{45}$$

and

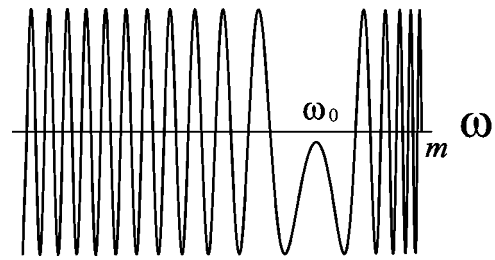

FIG. 2. The schematic behaviors of the integrand $\tilde{G}(r_*, r'_*; t)e^{-i\omega t}$ near $\omega \simeq m$ in Eq. (32). The integrand includes rapidly oscillating terms of $e^{2i\pi\kappa}$ and $e^{-i\omega t}$, but the phase of oscillation is stationary at $\omega = \omega_0$.

$$\gamma_\pm = \frac{\Gamma(2\lambda)^2\Gamma(1-4i\omega M)}{\Gamma(1/2+\lambda-2i\omega M)^2}(4m^2M^2)^{-\lambda}e^{i\pi\lambda} \pm \frac{\Gamma(-2\lambda)^2\Gamma(1-4i\omega M)}{\Gamma(1/2-\lambda-2i\omega M)^2}(4m^2M^2)^{\lambda}e^{-i\pi\lambda} \tag{46}$$

in the limit of $\kappa \to \infty$ [see (39) and (40) in [15]]. Then the following inequalities

$$|\eta_\pm| \gtrless |\gamma_\pm| \tag{47}$$

are satisfied when $\omega \gtrless 0$ respectively [see (43) in [15]]. When the inequalities (47) hold, the term $(ap_+ + bp_-)/(ap_+ - bp_-)$ can be expressed as the product of the rapidly oscillating term $e^{\pm 2i\pi\kappa}$ by $e^{i\varphi}$, where φ remains in the range $0 \leq \varphi < 2\pi$ even if κ becomes very large. The integrand (32) includes rapidly oscillating terms of $e^{2i\pi\kappa}$ and $e^{-i\omega t}$, which means physically that the scalar waves have multiple phases owing to the backscattering by the spacetime curvature, and the contribution from these waves are canceled by those with the inverse phase, unless the phase of oscillation becomes stationary, i.e.,

$$\frac{d}{d\omega}(\omega t \mp 2\pi\kappa) = 0 \tag{48}$$

for $\omega \gtrless 0$ respectively. We denote ω satisfying Eq. (48) by ω_0. Then, particular waves with the frequency ω_0 remain without cancellation, and contribute dominantly to the tail behaviors (see Fig. 2). In the limit of $|\omega| \to m$ we obtain the solutions of (48) as

$$t \simeq \pm \frac{2\pi\omega_0 m^2 M}{(m^2-\omega_0^2)^{3/2}} \tag{49}$$

for $\omega \gtrless 0$ respectively, or equivalently

$$\varpi_0 \equiv \sqrt{m^2-\omega_0^2} \simeq m\left(\frac{2\pi M}{t}\right)^{1/3}. \tag{50}$$

Approximating the integration (32) by the contribution from the close vicinity of ω_0, we obtain

$$G^C(r_*,r'_*;t)\simeq\frac{m}{4\sqrt{3}\lambda jk}(2\pi)^{5/6}(mM)^{1/3}(mt)^{-5/6}\times\sin(mt+\phi)\tilde{\psi}_1(r_*,m)\tilde{\psi}_1(r'_*,m). \tag{51}$$

Thus we can confirm that the decay law of $t^{-5/6}$ is a result of wave evolution in far distant regions $r\gg M$. We also find the existence of a phase shift ϕ given by

$$\phi=-\frac{3}{2}(2\pi mM)^{2/3}(mt)^{1/3}-\varphi(\varpi_0)+\frac{3}{4}\pi, \tag{52}$$

which modulates the basic oscillation with the period of $2\pi/m$ (see [15]). The multiple moment l and metric component $h(r)$ can affect only this modulation term in the asymptotically late-time evolution.

Note that if the conditions (47) break down, i.e., either

$$|\eta_\pm|\lesseqgtr|\gamma_\pm|, \tag{53}$$

for $\omega\gtrless0$, respectively, or

$$|\eta_\pm|=|\gamma_\pm| \tag{54}$$

are satisfied, all the contributions from scalar waves will be canceled more effectively. Then the tail with the decay rate of $t^{-5/6}$ cannot survive. Therefore, the conditions (47) are necessary for the tail to dominate. The physical implication is given in the next subsection.

C. Physical interpretation of the condition for the tail generation

It is easy to see that the inequalities (47) are satisfied for the background spacetimes discussed in previous papers [see (63) in [14], and (43) and (74) in [15]], in which the tail with the decay rate of $t^{-5/6}$ can dominate at very late times. Now we give the physical interpretation of Eq. (47) to discuss which background spacetime allows the development of the late-time tail.

Because λ^2 in Eq. (20) is real, λ should be either real or purely imaginary. From the expression (21) in Reissner-Nordström background, we find that the small mass ($mM\ll1$) gives a real λ, while the large mass ($mM\gg1$) gives an imaginary λ. The two inequalities (47) for $\omega\gtrless0$ can be reduced to

$$\frac{i}{\lambda}(ab^*-a^*b)\gtrless0 \tag{55}$$

when λ is real, and

$$\frac{1}{\gamma}(|b|^2-|a|^2)\gtrless0 \tag{56}$$

when $\lambda(=i\gamma)$ is purely imaginary, respectively.

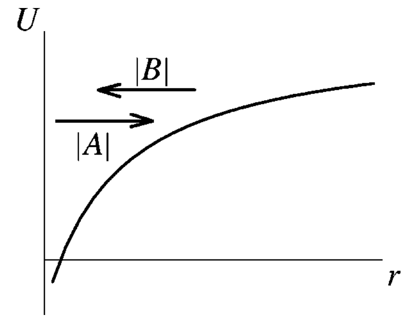

FIG. 3. The schematic behaviors of the effective potential U in Eq. (20) and wave mode $\tilde{\psi}_1$ in Eq. (58) in the far region where all of the conditions (16), (36) and (57) are satisfied.

One may claim that the condition (47) changes according to the value of λ. However, we can give the unified interpretation, independent of the value of λ, by paying attention to $\tilde{\psi}_1$ in the region

$$\omega r\ll\frac{1}{\kappa} \tag{57}$$

in addition to Eq. (36). When all of the conditions (16), (36) and (57) are satisfied, the second term which represents the Newtonian part in the effective potential (20) becomes dominant, compared with the other terms. In this region $\tilde{\psi}_1$ is approximated by

$$\tilde{\psi}_1\simeq Ae^{i2\sqrt{\kappa x}}+Be^{-i2\sqrt{\kappa x}}, \tag{58}$$

where A and B are

$$A=\pi^{-1/2}\kappa^{-1/4}x^{1/4}(2\lambda)e^{-i\pi/4}\{\kappa^{-\lambda}a\Gamma(2\lambda)e^{-i\pi\lambda}-\kappa^{\lambda}b\Gamma(-2\lambda)e^{i\pi\lambda}\} \tag{59}$$

and

$$B=\pi^{-1/2}\kappa^{-1/4}x^{1/4}(2\lambda)e^{-i\pi/4}\{\kappa^{-\lambda}a\Gamma(2\lambda)e^{i\pi\lambda}-\kappa^{\lambda}b\Gamma(-2\lambda)e^{-i\pi\lambda}\}, \tag{60}$$

respectively (see Fig. 3). In this region, independent of λ, the mode clearly shows a wave behavior with the amplitudes $|A|$ and $|B|$ corresponding to the outgoing and ingoing parts for $\omega>0$, while $|A|$ and $|B|$ correspond to the ingoing and outgoing parts for $\omega<0$. The difference between $|A|^2$ and $|B|^2$ is

$$|B|^2-|A|^2=2i\lambda\pi(ab^*-a^*b) \tag{61}$$

when λ is real. The conditions (55) under which the rapid oscillation of $e^{2i\pi\kappa}$ survives are equivalent with the inequalities

$$|B|\gtrless|A| \tag{62}$$

for $\omega\gtrless0$ respectively. On the other hand, when $\lambda(=i\gamma)$ is purely imaginary, we have

$$|B|^2-|A|^2=|2\lambda|^2|\Gamma(2\lambda)|^2(|b|^2-|a|^2)(e^{-2i\pi\lambda}-e^{2i\pi\lambda})$$

$$=8\pi\gamma(|b|^2-|a|^2), \tag{63}$$

which means the conditions (56) are also equivalent with the inequalities (62). Therefore, it is sufficient to consider the inequalities (62) independent of λ, as the conditions for the tail with the decay rate of $t^{-5/6}$ to dominate at late times. Equation (62) means *the amplitude of ingoing waves for* $\tilde{\psi}_1$ *is larger than that of outgoing waves*, in the region where Eqs. (16), (36) and (57) are all satisfied.

The origin of the slowly decaying tail as $t^{-5/6}$ of a massive scalar field can be considered a resonance by cooperation between dispersion and backscattering. It is a common feature when the scalar field has a nonzero mass that in far distant regions the effective potential (20) is a monotonously increasing function with r and the radial mode shows a wave behavior. If the central object is a black hole, the conditions (62) are surely satisfied because of the existence of the event horizon. So, we can conclude that this long-lived oscillating tail is generally observed in arbitrary spherical symmetric black-hole spacetimes.

IV. DISCUSSION

We have found that whether or not the tail with the decay rate of $t^{-5/6}$ develops at very late times can be judged relevantly by wave modes only in far distant regions. Then, even when the central object is a rotating black hole, only the parameters M' and λ in the effective potential (20) will be changed in far distant regions. Strictly, background spacetimes in this paper are limited to the class of static and spherically symmetric. However, since these are not relevant to the conditions (62), the same tail behaviors are expected to dominate also in Kerr spacetimes.

We compare our analytical result with their numerical simulation [12]. As far as the intermediate late-time behavior is concerned, our result agrees with [12]. However they claimed "SI perturbation fields decay at late times slower than any power law" in [12], which disagrees with our present result and previous ones [14,15] that the late-time tail of a massive scalar field is a power law with index $-5/6$. We believe that the integration time in [12] is too short to find the true asymptotic behavior.

Now we remark that the region of spacetimes where the tail with the decay rate of $t^{-5/6}$ dominates is limited. This feature can be understood by considering the behavior of $\tilde{\psi}_1$. In the region

$$\varpi_0 r \gtrsim \kappa(\omega_0), \tag{64}$$

$\tilde{\psi}_1$ is reduced to

$$\tilde{\psi}_1 \simeq \sqrt{\frac{2}{\pi}}\lambda\left\{\left(\frac{\kappa}{2e\varpi}\right)^{\kappa} e^{\varpi r-\kappa\ln r}(\eta_- e^{i\pi\kappa}+\gamma_- e^{-i\pi\kappa}) + \left(\frac{\kappa}{2e\varpi}\right)^{-\kappa} e^{-\varpi r+\kappa\ln r} e^{i\pi/2-i\pi\kappa}\gamma_-\right\} \tag{65}$$

and the saddle point at $\omega=\omega_0$ (50) has disappeared because of the terms of $e^{\varpi r-\kappa\ln r}$ which change exponentially. Therefore it is obvious that the $t^{-5/6}$ tail dominates only within the region

$$r \ll M^{1/3} t^{2/3}. \tag{66}$$

What kind of behavior dominates in the region $r \gg M^{1/3}t^{2/3}$, in particular, near the null cone $r\to t$? Now we must find saddle points as solutions of the following equation

$$\frac{\partial}{\partial\omega}(-i\omega t+\varpi r-\kappa\ln r)=0, \tag{67}$$

instead of Eq. (48). In general, solutions of Eq. (67) are complex functions of t and r. However we can find a simple asymptotic solution as

$$\omega_1 \simeq \frac{imt}{\sqrt{t^2-\tilde{r}^2}}, \tag{68}$$

for high frequency $\omega_1 \gg m$, which is compatible with the limit of

$$\tilde{r} \to t, \tag{69}$$

where $\tilde{r}$ is

$$\tilde{r} \equiv r+(M+M')\ln r, \tag{70}$$

which is modified due to red shift. The expression (68) means that we can calculate Green's function using the saddle point integration by deforming the integration contour into the straight line AOB in Fig. 1. When ω_1 is a large value, considering the region $2\omega_1 r' \gg 1$ also, Green's function is reduced to

$$G(r_*, r'_*; t) \sim \int -\frac{e^{-i\omega t}}{2\varpi}(e^{-\varpi R}+e^{\varpi R})d\omega, \tag{71}$$

and we obtain

$$\omega_1 \simeq \frac{imt}{\sqrt{t^2-R^2}} \tag{72}$$

as the saddle point rather than Eq. (68), where R is

$$R = \tilde{r} - \tilde{r}'. \tag{73}$$

Approximating the integration (12) by the contribution from the immediate vicinity of ω_1, we obtain

$$G(r_*,r'_*;t)\sim\left|\frac{e^{i(-\omega t+\sqrt{\omega^2-m^2}R)}}{2i\sqrt{\omega^2-m^2}}\right|_{\omega=\omega_1}\int d\omega$$
$$\times\exp\left[i\left|\frac{\partial^2}{\partial\omega^2}(-\omega t+\sqrt{\omega^2-m^2}R)\right|_{\omega=\omega_1}\times(\omega-\omega_1)^2\right]$$
$$=e^{i(-\omega_1 t+\sqrt{\omega_1^2-m^2}R)}2^{-3/4}m^{-1/2}\times(t+R)^{-1/4}(t-R)^{-1/4}. \quad (74)$$

The radial part of the scalar field Φ near null cone $R\simeq t$, together with the geometrical factor $1/r$, behaves as

$$\frac{\psi}{r}\sim e^{-im(2tu)^{1/2}}2^{-3/4}m^{-1/2}t^{-5/4}u^{-1/4}, \quad (75)$$

where u is

$$u=t-R. \quad (76)$$

This behavior (75) is similar to the case of Minkowski spacetimes, except for $\tilde{r}$ including red shift factor, instead of r. Massive fields near the null cone decay more rapidly than $t^{-5/6}$.

Finally we comment about late-time tail behaviors when the central object is a normal star such as a neutron star or a boson star. If the expression of $\tilde{\psi}_1$ in Eq. (28) is assumed to be extended to the region $r\leqslant M$, then we must require $\tilde{\psi}_1$ to be regular at $r=0$. This leads to the equality $|A|=|B|$ which means that the amplitude of outgoing is equivalent to that of ingoing. Then the tail with the decay rate of $t^{-5/6}$ never develops. Though this extension of Eq. (28) may not be valid, we can expect the equality $|A|=|B|$ to be valid, unless some absorption of waves occurs in the inner region. This is a future problem to be checked by giving a background gravitational field with a regular center.

ACKNOWLEDGMENTS

The authors would like to thank Andrei V. Frolov for valuable comments and discussions.

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